Module: Mathematics 3 L2 (Electronics, Automation, Telecommunication) **Exam Solution**

Partie 01 : Exercise 1 (5 pts)

Calculate the volume:

$$z = \sqrt{x^2+y^2} \leq R \Rightarrow \begin{cases} x^2+y^2 = r^2 \leq R^2 \\ z = \sqrt{x^2+y^2} = \sqrt{r^2} \\ dxdy = r.dr.d\theta \end{cases} \Rightarrow \begin{cases} r \leq R \text{ et } \theta \in [0, 2\pi] \\ z = r \\ dxdy = r.dr.d\theta \end{cases}$$

$$V = \int_0^{2\pi}\int_0^R r.z.dr\,d\theta = \int_0^{2\pi}\int_0^R r^2 dr\,d\theta = \int_0^{2\pi} d\theta.\int_0^R r^2 dr = \theta]_0^{2\pi}.\frac{r^3}{3}\Big]_0^R$$

$$V = (2\pi - 0).\frac{(R^3-0)}{3}$$

$$V = .\frac{2\pi R^3}{3}$$

Exercise 2 (4 pts)

1. *Study the convergence :*

a. $S1 = \sum_{n=0}^{\infty} n.e^{-3n} \rightarrow \lim_{n\to\infty} n.e^{-3n} = 0$

According to the d'Alembert criterion: $\lim_{n\to\infty}\frac{U_{n+1}}{U_n} = \lim_{n\to\infty}\frac{(n+1).e^{-3(n+1)}}{n.e^{-3n}} = e^{-3} < 1$

So the S1 serie CONVERGE.

b. $S2 = \sum_{n=0}^{\infty} e^n \rightarrow \lim_{n\to\infty} e^n = +\infty$ $\Rightarrow$ *So the S2 serie DIVERGE*

2. *Find the radius of convergence :* $S3 = \sum_{n=0}^{\infty} 4.sin[(3x)^n]$.

According to the d'Alembert criterion: $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{4.sin[(3x)^{n+1}]}{4.sin[(3x)^n]}\right| < 1$

$$\lim_{n\to\infty}\left|\frac{4.sin[(3x)^{n+1}]}{4.sin[(3x)^n]}.\frac{(3x)^{n+1}}{(3x)^{n+1}}\frac{(3x)^n}{(3x)^n}\right| \lim_{n\to\infty}\left|\frac{sin[(3x)^{n+1}]/(3x)^{n+1}}{sin[(3x)^n]/(3x)^n}.\frac{(3x)^{n+1}}{(3x)^n}\right| = 3|x| < 1$$

So we get : $|x| < \frac{1}{3} = R$ …… *the radius of convergence* $R=\frac{1}{3}$

Partie 02 :

Exercise 3 (6 pts)

- *Solve the following differential equation*

$$y'' + 4y = 4.\sin(2.x) + 3.e^{2x}$$

$$: y = y_0 + y_p \Rightarrow \begin{cases} y_0'' + 4y_0 = 0 \\ y_p'' + 4y_p = 4.\sin(2.x) + 3.e^{2x} \end{cases} \text{We pose } y_p = k(x).y_0$$

$$y_0'' + 4y_0 = 0 \Rightarrow \text{We pose } y = e^{r.x}$$

We obtain $e^{rx}[r^2 + 4] = 0 \Rightarrow \begin{cases} r_1 = 2i \\ r_2 = -2i \end{cases} \Rightarrow$

$$y_0 = c_1.e^{2ix} + c_2.e^{-2ix} = A.\cos(2x) + B.\sin(2x)$$

So $y_p = k(x).e^{2ix}$

$$y_p' = k'.e^{2ix} + 2i.k.e^{2ix} \Rightarrow y''_p = k''.\cos(2x) + 4i.k'.e^{2ix} - 4.k.e^{2ix}$$

$$y_p'' + 4y_p = 4.\sin(2.x) + 3.e^{2x} = k''.e^{2ix} + 4ik'e^{2ix}$$

$$4e^{-2ix}\left(\frac{e^{2ix} - ie^{-2ix}}{2i}\right) + 3.e^{2x(1-i)} = k'' + 4ik'$$

$$k_0'' + 4i.k'_0 = 0 \Rightarrow k_0' = \mathrm{d}.e^{-4ix} \Rightarrow k_p'' = \mathrm{d}'.e^{-4ix} - 4i.d.e^{-4ix}$$

$$k'' + 4i.k' = \mathrm{d}'.e^{-4ix} = 4e^{-2ix}\left(\frac{e^{2ix} - ie^{-2ix}}{2i}\right) + 3.e^{2x(1-i)}$$

$$\mathrm{d}' = -2i.e^{4ix} - 2 + 3.e^{2x(1+i)} \Rightarrow \mathrm{d} = -\frac{e^{4ix}}{2} - 2x + \frac{3}{2(i+1)}.e^{2x(1+i)}$$

$$k_0' = \mathrm{d}.e^{-4ix} = \left(-\frac{e^{4ix}}{2} - 2x + \frac{3}{2(i+1)}.e^{2x(1+i)}\right)e^{-4ix}$$

$$k_0' = -\frac{1}{2} - 2x.e^{-4ix} + \frac{3}{2(i+1)}.e^{2x(1-i)}$$

$$k_0 = \int\left(-\frac{1}{2} - 2x.e^{-4ix} + \frac{3}{2(i+1)}.e^{2x(1-i)}\right)dx$$

$$k_0 = \frac{-x}{2} - \int(2x.e^{-4ix})dx + \frac{3}{4(i+1)(i-1)}.e^{2x(1-i)}$$

$$\int(2x.e^{-4ix})dx = 2x\frac{e^{-4ix}}{-4i} - \int\frac{e^{-4ix}}{2i}dx = i.x\frac{e^{-4ix}}{2} + \frac{e^{-4ix}}{8}$$

So $k_0 = \frac{-x}{2} + i.x\frac{e^{-4ix}}{2} + \frac{e^{-4ix}}{8} + \frac{3}{4(i+1)(i-1)}.e^{2x(1-i)} + c$

$$y = \left(\frac{-x}{2} + i.x\frac{e^{-4ix}}{2} + \frac{e^{-4ix}}{8} + \frac{3}{4(i+1)(i-1)}.e^{2x(1-i)} + c\right).e^{2ix}$$

$$y = -x\left(\frac{e^{2ix} - e^{-2ix}}{2}\right) + \frac{e^{-2ix}}{8} + ce^{2ix} + \frac{3}{8}.e^{2x}$$

$$y = (A - x).\sin(2x) + B.\cos(2x) + \frac{3}{8}.e^{2x}$$

Exercise 4 (5 pts)

- *Expand the equation according to the Fourier series :*

$$f(x) = e^{2x} - e^{-2x} \quad : \quad x \in [-1, 1] \ldots\ldots\ldots. T = 2 \Rightarrow w = \frac{2.\pi}{T} = \pi$$

$$f(-x) = -f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ the function f(x) odd}$$

$$\Rightarrow a_n = 0 \qquad \forall n$$

$$f(x) = \sum_{n=0}^{\infty} [a_n cos(nwx) + b_n sin(nwx)] = \sum_{n=1}^{\infty} b_n sin(nwx)$$

$$b_n = \frac{2}{2}\int_0^1 f(x).\sin(n\pi x)dx = \int_0^1 e^{2x}.\sin(n\pi x)dx - \int_0^1 e^{-2x}.\sin(n\pi x)dx$$

$$b_n = \frac{e^{2x}}{2}.\sin(n\pi x)\Big|_0^1 - \int_0^1 n\pi\frac{e^{2x}}{2}.\cos(n\pi x)\,dx + \frac{e^{-2x}}{2}.\sin(n\pi x)\Big|_0^1 - \int_0^1 n\pi\frac{e^{-2x}}{2}.cos(n\pi x)dx$$

$$b_n = \frac{-n\pi e^{2x}}{4}.\cos(n\pi x)\Big|_0^1 - \int_0^1 \frac{(n\pi)^2 e^{2x}}{4}.\sin(n\pi x)\,dx + \frac{n\pi e^{-2x}}{4}.\cos(n\pi x)\Big|_0^1 + \int_0^1 \frac{(n\pi)^2 e^{-2x}}{4}.\sin(n\pi x)\,dx$$

$$b_n = \frac{-n\pi e^{2x}}{4}.\cos(n\pi x)\Big|_0^1 + \frac{n\pi e^{-2x}}{4}.\cos(n\pi x)\Big|_0^1 - \frac{(n\pi)^2}{4}b_n$$

$$(1 + \frac{(n\pi)^2}{4})b_n = \frac{-n\pi e^2}{4}.\cos(n\pi) + \frac{n\pi}{4} + \frac{n\pi e^{-2}}{4}.\cos(n\pi) - \frac{n\pi}{4} = \frac{n\pi\cos(n\pi)}{4}(e^{-2} - e^2).$$

$$b_n = \frac{n\pi\cos(n\pi)}{4 + (n\pi)^2}(e^{-2} - e^2).$$

$$b_n = \frac{n\pi(-1)^n}{4 + (n\pi)^2}(e^{-2} - e^2).$$

$$f(x) = \pi(e^{-2} - e^2)\sum_{n=1}^{\infty} \frac{n(-1)^n}{4 + (n\pi)^2} sin(n\pi x)$$