University of Kasdi Merbah Ouargla

Faculty of New Technologies of Information and Communication

Computer Science and Information Technolgy Department

First year Engineer of Computer Science **Module Analysis 1**

Final Exam (2023/2024)

Exercise 1 (4p). *Let* $A = \left\{1 - \frac{2(-1)^n}{n}, \quad n \in \mathbb{N}^*\right\}$. *Answer true or false with explanation*

1. A is bounded. *2.* $\sup A = 7$. *3.* $\max A$ *not exist.* *4.* $\min A = 1$.

Exercise 2 (4p). *From the equivalent between the functions calculate the limit, when* x *tends to* 0*, of:*

$$f(x) = \frac{(1-\cos x)\sin x}{x^2 \ln(1+x)}$$

Exercise 3 (6p). *Let the function* f *define by*

$$f(x) = \begin{cases} \dfrac{3-x^2}{2} & if\ x < 1 \\ \dfrac{1}{x} & if\ x \geq 1 \end{cases}$$

1. Determine the domain of definition $\mathcal{D}_f$ *of the function* f.

2. Show that f *is continuous on* $\mathcal{D}_f$.

3. Prove that f *is differentiable on* $\mathcal{D}_f$.

4. By applying the finite increment theorem, show that there exists $c \in]0, 2[$ *such that* $2f'(c) = f(2) - f(0)$

- Determine all possible values of c

Exercise 4 (6p). *We consider the sequences* $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *defined for all* $n \in \mathbb{N}^*$ *by*

$$u_n = \sum_{k=1}^{k=n} \frac{1}{k^2}, \quad v_n = u_n + \frac{3}{n}$$

1. Study the monotony of the sequences $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$. *(3p)*

2. Show that for all $n \in \mathbb{N}^*$, $u_n \leq v_n$. *(1p)*

3. *Prove that the sequence* $(v_n - u_n)_{n\in\mathbb{N}^*}$ *converges to* 0. *(1p)*
4. *What have we just shown about the sequences* $(u_n)_n \in \mathbb{N}^*$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *(1p)?*

Good luck

Correction

Correction 1. *We have* $A=\left\{1-\dfrac{2(-1)^n}{n}, \quad n\in\mathbb{N}^*\right\}$ *and for all* $n\in\mathbb{N}$ *we have*

$$A=\begin{cases}1-\dfrac{2}{n}, & if\ n\ even\\ 1+\dfrac{2}{n}, & if\ n\ odd.\end{cases}$$

For $n=1$ *we get* $A=3$*, for* $n=2$ *we have* $A=0$ *and when* $n\longrightarrow+\infty$ *we get* $A=1$*, then for all* $n\geq 1$ *we have* $0\leq A\leq 3$.

1. A *is bounded. True because* $0\leq A\leq 3$

2. $\sup A=7$*. False,* $\sup A=3$.

3. $\max A$ *not exist. False, exist and* $\max A=3$.

4. $\min A=1$*. False,* $\min A=0$.

Correction 2. *Calculate the limit, when* x *tends to* 0*, of:*

$$f(x)=\frac{(1-\cos x)\sin x}{x^2\ln(1+x)}$$

We know that: $1-\cos x\sim_0\dfrac{x^2}{2}$, $\sin x\sim_0 x$ *and* $\ln(1+x)\sim_0 x$ *then*

$$f(x)\sim_0\frac{\frac{x^2}{2}x}{x^3}=\frac{1}{2}$$

so $\lim_{x\longrightarrow 0}f(x)=\dfrac{1}{2}$.

Correction 3. *1.* $\mathcal{D}_f=\mathbb{R}$.

2.- For $x<1$*, the function* f *is continuous because* f *polynome.*

- For $x>1$*, the function* $\dfrac{1}{x}$ *is continuous.*

- For x=1 we have

$$\lim_{x\longrightarrow 1^+}f(x)=\lim_{x\longrightarrow 1^-}f(x)=f(1)=1.$$

Then f *is continuous for all* $x\in\mathcal{D}_f$.

3. We note that for $x\neq 1$*,* f *is differentiable.*

-For $x\longrightarrow 1^-$ *we have:*

$$\lim_{x\longrightarrow 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\longrightarrow 1^-}\frac{\frac{3-x^2}{2}-1}{x-1}=-1$$

-For $x \longrightarrow 1^+$ we have:

$$\lim_{x\longrightarrow 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\longrightarrow 1^+} \frac{\frac{1}{x}-1}{x-1} = -1$$

We conclude that $f'_r(1) = f'_l(1)$ and f is differentiable on 1 so is differentiable on $\mathcal{D}_f$.

4. We have f is continuous and differentiable on $\mathbb{R}$ then it is continuous and differentiable on $[0,2]$ so we can applying the finite increment theorem and there exist $c \in]0,2[$ such that $f(2)-f(0) = 2f'(c)$ so

$$\frac{1}{2} - \frac{3}{2} = 2f'(c) \Leftrightarrow f'(c) = -\frac{1}{2}$$

the possible values of c:

- For $0 < c \leq 1$, $f'(c) = -c = -\frac{1}{2} \Rightarrow c = \frac{1}{2}$.

- For $1 < 1 \leq 2$, $f'(c) = -\frac{1}{c^2} = -\frac{1}{2} \Rightarrow c^2 = 2 \Rightarrow c = \pm\sqrt{2}$.

Since $-\sqrt{2} \notin]0,2[$ so $c = \sqrt{2}$

Correction 4. *1. We have*

$$u_{n+1} - u_n = \sum_{k=1}^{k=n+1} \frac{1}{k^2} - \sum_{k=1}^{k=n} \frac{1}{k^2} = \frac{1}{(n+1)^2} > 0$$

then, for all $n \geq 1$ we have $(u_n)_{n\in\mathbb{N}^*}$ increasing so it's monotone.

Another hand, we have

$$\begin{aligned} v_{n+1} - v_n &= \sum_{k=1}^{k=n+1} \frac{1}{k^2} + \frac{3}{n+1} - \sum_{k=1}^{k=n} \frac{1}{k^2} - \frac{3}{n} \\ &= \frac{1}{(n+1)^2} - \frac{3}{n(n+1)} \\ &= \frac{-2n^2-5n-3}{n(n+1)^3} < 0 \end{aligned}$$

then, for all $n \geq 1$ we have $(v_n)_{n\in\mathbb{N}^*}$ decreasing so it's monotone.

2. For all $n \in \mathbb{N}^*$, we have

$$u_n - v_n = -\frac{1}{n} < 0 \Leftrightarrow u_n \leq v_n.$$

3. We have

$$v_n - u_n = \frac{1}{n}, \quad \forall n \geq 1 \Leftrightarrow \lim_{n\longrightarrow+\infty} v_n - u_n = \lim_{n\longrightarrow+\infty} \frac{1}{n} = 0.$$

4. We conclude that $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *are adjacent.*