MIDTERM EXAM CORRECTION

Exercise 1. **(6 pts)** Consider the following system:

$$\begin{cases} x - 2y + kz = -k \\ y + z = k \\ (k+2)x + 2ky - z = 1 \end{cases}$$

1. For which values of k does the system have a unique solution?
2. For which values of k does the system have infinitely many solutions? If any, find for those values of k all system solutions.

Solution
First, we need to write the augmented matrix $(\mathbf{A}|\mathbf{b})$ in an echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & k & -k \\ 0 & 1 & 1 & k \\ k+2 & 2k & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & k & -k \\ 0 & 1 & 1 & k \\ 0 & 4k+4 & -(k+1)^2 & (k+1)^2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & -k \\ 0 & 1 & 1 & k \\ 0 & 0 & -(k+1)(k+5) & (k+1)(-3k+1) \end{array}\right) \text{(1pt)}$$

1. We can notice that if $k \neq -1$ and $k \neq -5$, the matrix is full rank. which means that in that case, the system admits a unique solution (1pt).
2. If $k = -1$ that the system has an infinite number of solutions (1pt). In that case, the echelon form of the augmented matrix is as follows:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A particular solution is obtained by setting $z = 0$, we obtain then the solution $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (1pt).

The general solution can be obtained by solving the system $\mathbf{Ax} = \mathbf{0}$. We write that matrix $\mathbf{A}$ in a reduced echelon form (1pt):

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

By using the (-1)-trick to the zero-row free reduced echelon form, we obtain:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Therefore, the set of all solutions of the linear system is

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \text{(1pt)}$$

Exercise 2. **(6 pts)** Let $V = \textbf{span}(v1; v2; v3) \subset \mathbb{R}^4$ where

$$v_1 = (1, 1, 1, 1)^\top, \quad v_2 = (2, 2, 2, 3)^\top, \quad v_3 = (1, 1, 2, -3)^\top$$

Let W be the subset of V which contains all and only the vectors of V that have the first two components equal to 0.

1. Compute $\dim V$.
2. Give a description of the set W. Is W a subspace of V?
3. Find a basis for W.

Solution

1. Consider the matrix **A** which columns are the vectors v_1, v_2, v_3:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & -3 \end{pmatrix}$$

Writing the echelon form of the matrix, we obtain

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We can notice that vectors v_1, v_2, v_3 are linearly independent and thus they constitute a basis for V, which means that $\dim V = 3$ (1pt).

2. W is the subset of V which contains all and only the vectors of V that have the first two components equal to 0. Therefore, we can write elements of W as follows

$$W = \left\{ (x_1, x_2, x_3, x_4)^\top \in V : x_1 = x_2 = 0 \right\} \text{ (1pt)}$$

We can verify that W is a subspace in V, since

- $\mathbf{0} = (0, 0, 0, 0)^\top \in W$ (0.5pt),

- For $w_1, w_2 \in W$, we have

$$w_1 = \begin{pmatrix} 0 \\ 0 \\ x \\ y \end{pmatrix} \quad w_2 = \begin{pmatrix} 0 \\ 0 \\ z \\ t \end{pmatrix} \Rightarrow w_1 + w_2 = \begin{pmatrix} 0 \\ 0 \\ x+z \\ y+t \end{pmatrix} \Rightarrow w_1 + w_2 \in W \text{ (0.75pt)}$$

- For $w \in W$, we have

$$w = \begin{pmatrix} 0 \\ 0 \\ x \\ y \end{pmatrix} \Rightarrow \lambda w = \begin{pmatrix} 0 \\ 0 \\ \lambda x \\ \lambda y \end{pmatrix} \Rightarrow \lambda w \in W \text{ (0.75pt)}$$

3. The set W can be written as $W = V \cap Z$, where $Z = \{(x, y, z, t) \in \mathbb{R}^4 : x = y = 0\}$. This means that Z is the set of solutions of the linear system

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

On the other hand, elements of the vector space V can be described as a linear combination of vectors v_1, v_2, v_3. Thus they are all vectors v that can be written as $\mathbf{Ax} = \mathbf{v}$. considering the augmented matrix $\mathbf{A}|\mathbf{v}$, we have

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & v_1 \\ 1 & 2 & 1 & v_2 \\ 1 & 2 & 2 & v_3 \\ 1 & 3 & -3 & v_4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & v_1 \\ 0 & 0 & 0 & v_2 - v_1 \\ 0 & 0 & 1 & v_3 - v_1 \\ 0 & 1 & -4 & v_4 - v_1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & v_1 \\ 0 & 1 & -4 & v_4 - v_1 \\ 0 & 0 & 1 & v_3 - v_1 \\ 0 & 0 & 0 & v_2 - v_1 \end{array}\right)$$

Thus V can be described as the solution for the linear system $v_2 - v_1 = 0$. The set W is then the set of solutions of the linear system

$$\begin{cases} x - y = 0 \\ x = 0 \\ y = 0 \end{cases}$$

We can notice that the first equation is redundant, thus $W = V \cap Z = Z$. Therefore, a basis for W is $\left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]$ (2pts)

Exercise 3. **(2 pts)** Consider the following definition: If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2, denoted by $S_1 + S_2$, is the set $\{x + y : x \in S_1, y \in S_2\}$. Now consider two subspaces W_1 and W_2 of V.

- Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

Solution

First, consider $u \in W_1$. Since W_2 is a subspace of V then $0 \in W_2$. This means that $u + 0 = u \in W_1 + W_2$ and consequently, $W_1 \subset W_1 + W_2$.

Similarly, with $v \in W_2$ and $0 \in W_2$, we have $0 + v = v \in W_1 + W_2$. Thus $W_2 \subset W_1 + W_2$. (0.5pt)

Now, let us prove that $W = W_1 + W_2$ is a subspace (1.5pt):

- $0 \in W_1 \subset W$.
- For two vectors w and w' from W, we have $w = w_1 + w_2$ and $w' = w_1' + w_2'$. Thus, $w + w' = w_1 + w_2 + w_1' + w_2' = (w_1 + w_1') + (w_2 + w_2') \in W_1 + W_2 = W$
- For a vector $w \in W$ we have $\lambda w = \lambda(w_1 + w_2) = (\lambda w_1) + (\lambda w_2) \in W_1 + W_2 = W$.

Exercise 4. **(6 pts)** Consider the Euclidean vector space $\mathbb{R}^3$ with the dot product. Consider the subspace defined by $U = \mathbf{span}(v_1; v_2) \subset \mathbb{R}^3$, such that

$$v_1 = (-1, 1, 1)^\top; v_2 = (2, -1, -2)^\top$$

Let $v = (8, 4, 16)^\top$.

1. Show that $v \notin U$.
2. Use Gram-Schmidt orthogonalization to transform the matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 1 & -2 \end{bmatrix}$ into a matrix B with orthonormal columns.
3. Determine the orthogonal projection of v onto U then compute the distance $d(v, U)$.

Solution

1. To show that $v \notin U$, we show that the matrix $\mathbf{A}$ whose columns are v_1, v_2 and v is of full rank (this means that v cannot be written as a linear combination of v_1 and v_2). For this, we need to write $\mathbf{A}$ in a echelon form:
$$\begin{pmatrix} -1 & 2 & 8 \\ 1 & -1 & 4 \\ 1 & -2 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -8 \\ 0 & 1 & 12 \\ 0 & 0 & 24 \end{pmatrix} \text{ (1pt)}$$
Thus, $\mathbf{rk}(\mathbf{A}) = 3$, which means that v, v_1, v_2 are linearly independent and therefore $v \notin U$.

2. Consider the vectors v_1 and v_2. We use the Gram Schmidt orthogonalization to obtain orthonormal vectors. In this case, the Gram-Schmidt process is described as follows:
$$\begin{cases} u_1 & = v_1 \\ u_2 & = v_2 - \pi_{u_1}(v_2) \end{cases}$$
We have then $u_1 = v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = v_2 - \pi_{u_1}(v_2)$. The projection matrix is defined by
$$\mathbf{P}_\pi = \frac{v_1 v_1^\top}{\|v_1\|^2} = \frac{1}{3}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \text{ (1pt)}$$
and
$$\pi_{u_1}(v_2) = \frac{1}{3}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix} \text{ (0.75pt)}$$

Thus

$$u_2 = v_2 - \pi_{u_1}(v_2) = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{(0.75pt)}$$

To obtain $\mathbf{B}$, we orthonormalize each one of the vectors u_1 and u_2:

$$\mathbf{B} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \quad \text{(0.5pt)}$$

3. We first compute the projection matrix for U. since vector columns in B constitute orthonormal basis of U we have

$$\mathbf{P}_\pi = \mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1}\mathbf{B}^\top = \mathbf{B}\mathbf{B}^\top$$

$$\mathbf{P}_\pi = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \quad \text{(1pt)}$$

The orthogonal projection of v onto U is the computed as follows:

$$\pi_U(v) = \mathbf{P}_\pi v = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 8 \\ 4 \\ 16 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix} \quad \text{(0.5pt)}$$

The distance $d(v, \pi_U(v))$ is then given by $\sqrt{(8+4)^2 + (4-4)^2 + (16-4)^2} = 12\sqrt{2}$. (0.5pt)