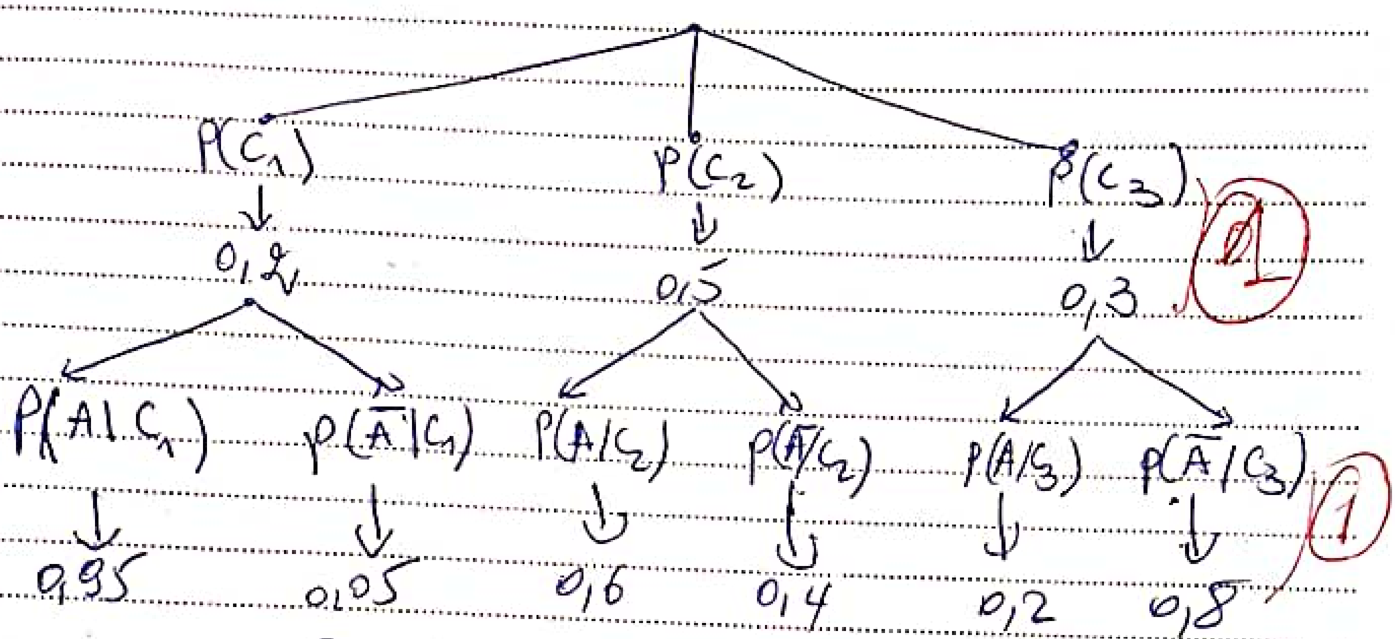
Correction of the written exam

Exercise 1 (06pts) : A: "the student passes the exam" (0.5)

1)



2) Using the law of the probabilities total

$$P(A) = \sum_{i=1}^{3} P(C_i) \cdot P(A/C_i)$$ (1)

$$= P(C_1) \cdot P(A/C_1) + P(C_2) \cdot P(A/C_2) + P(C_3) P(A/C_3)$$

$$= (0,2 \times 0,95) + (0,5 \times 0,6) + (0,3 \times 0,2)$$ (0.5)

$$P(A) = 0,55$$ (0.5)

3) Using Bayes law: (1)

$$P(C_3 / A) = \frac{P(A/C_3) P(C_3)}{P(A)} = \frac{0,3 \times 0,2}{0,55}$$

$$P(C_3 / A) = 0,109$$ (0.5)

Exercise 2: (6 pts)

1) The probability law of the r.v. X is: "Binomial" with $p = 0,01$ (0,5) and $n = 200$ (0,5)

$\Rightarrow \beta(200, 0,01)$ (0,5)

$P(X=k) = C_{200}^{k} (0,01)^{k} (0,99)^{-k}$ (0,5), $k = 0,1,\dots,200$.

(0,5)

2) We have $\begin{cases} n = 200 > 30 \\ n.p = 0,01 \times 200 = 2 \leq 5 \end{cases}$ (0,5)

Hence, we can approximate this probability law by a Poisson law with $\lambda = n.p = 2$

$\Rightarrow P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$ (0,5), $k = 0,1,\dots,200$

$\Rightarrow P(X=k) = e^{-2} \frac{2^k}{k!}$ (0,5)

3) $P(X=5) = e^{-2} \frac{2^5}{5!} \simeq 0,0361$ (0,5)

4) $P(X>4) = 1 - P(X \leq 4)$

$= 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right]$

$= 1 - e^{-2}\left[1 + 2 + 2 + \frac{4}{3} + \frac{2}{3}\right]$

$P(X>4) \simeq 0,0053$ (1,5)

Exercise 3 : (08 pts)

$$f_{XY}(x,y) = \begin{cases} k(x+y) & ; 0 \leq x \leq 2, 0 \leq y \leq 2 \\ 0 & ; \text{otherwise} \end{cases}$$

1) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1$

$$\Rightarrow \int_0^2\int_0^2 k(x+y)\,dx\,dy = 1$$

$$\Rightarrow k\int_0^2 \left[\frac{1}{2}x^2 + xy\right]_0^2 dy = 1$$

$$\Rightarrow k\int_0^2 (2+2y)\,dy = 1$$

$$\Rightarrow k\left[2y + y^2\right]_0^2 = 1$$

$$\Rightarrow k(4+4) = 1 \Rightarrow 8k = 1 \Rightarrow \boxed{k = \frac{1}{8}}$$ (1)

2) $f(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$

$$= \int_0^2 \frac{1}{8}(x+y)\,dy = \frac{1}{8}\left[xy + \frac{1}{2}y^2\right]_0^2$$

$$f_X(x) = \begin{cases} \frac{1}{4}(x+1) & ; 0 \leq x \leq 2 \\ 0 & ; \text{otherwise} \end{cases}$$ (0.5)

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \begin{cases} \frac{1}{4}(y+1) & ; 0 \leq y \leq 2 \\ 0 & ; \text{otherwise} \end{cases}$$ (0.5)

3. $P(X<1 \mid Y<1) = \dfrac{\int_{-\infty}^{1}\int_{-\infty}^{1} f_{XY}(x,y)\,dx\,dy}{\int_{-\infty}^{1} f_Y(y)\,dy}$

$$P(X<1 \mid Y<1) = \frac{\frac{1}{8}\int_0^1\int_0^1 (x+y)\,dx\,dy}{\frac{1}{4}\int_0^1 (y+1)\,dy} = \frac{1/8}{3/8}$$

$$\boxed{P(X<1 / Y<1) = 1/3}$$ 1

4) $E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \frac{1}{4}\int_0^2 (x+1)\,x\,dx$

$$= \frac{1}{4}\int_0^2 (x^2+x)\,dx$$

$$= \frac{1}{4}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^2 = \frac{7}{6}$$ 1

$E[Y] = E[X] = \frac{7}{6}$

$$E[XY] = \frac{1}{8}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\,[x+y]\,dx\,dy$$

$$= \frac{1}{8}\int_0^2\int_0^2 (x^2y + xy^2)\,dx\,dy$$

$$= \frac{1}{8}\int_0^2 \left[\frac{y}{3}x^3 + \frac{y^2}{2}x^2\right]_0^2 dy$$

$$= \frac{1}{8}\int_0^2 \left(\frac{8}{3}y + 2y^2\right) dy$$

$$= \frac{1}{8}\left[\frac{8}{6}y^2 + \frac{2}{3}y^3\right]_0^2$$

$$= \frac{1}{8}\left[\frac{32}{6} + \frac{16}{3}\right]$$

$$\boxed{E[XY] = \frac{4}{3}}$$ 1

$$\rho_{XY} = \frac{Cov(XY)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

$\sigma_X = \sqrt{V_X}$, $V_X = E[X^2] - E^2[X]$ 1

$E[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \frac{1}{4}\int_0^2 x^2(x+1)\,dx = \frac{5}{3}$

4

$E[X^2] = E[Y^2] = \frac{5}{3}$

$V_x = V_y = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{11}{36} \Rightarrow \sigma_x = \sigma_y = \frac{\sqrt{11}}{6}$ (1)

$\rho_{xy} = \frac{4/3 - \left(\frac{7}{6}\right)^2}{\frac{11}{36}} = \frac{\frac{4}{3} - \frac{49}{36}}{\frac{11}{36}} = \frac{48-49}{11}$

$\rho_{xy} = -1/11 \simeq -0,091$ (1)

5) $\rho_{xy} \neq 0 \Rightarrow$ Y and X are not independent. (1)

(4)